ELSEVIER Journal of Mathematical Economics 28 (1997) 470–479

JOURNAL OF Mathematical ECONOMICS

Limited arbitrage is necessary and sufficient for the existence of an equilibrium

Graciela Chichilnisky *

405 Low Memorial Library, Columbia University, New York, NY 10027, USA

Submitted May 1996; accepted August 1996

Abstract

In Chichilnisky (Working Paper No. 586, 1991), Chichilnisky (Working Paper No. 650, 1992) and Chichilnisky (*Economic Theory*, 1995, 5, 79–108), I introduced the concept of a global cone and used it to define a condition on endowments and preferences, 'limited arbitrage', which I showed to be necessary and sufficient for the existence of a competitive equilibrium. In response to a comment (Monteiro et al., *Journal of Mathematical Economics*, 1997, 26, 000-000), I show here that the authors misunderstood my results by focussing on brief announcements which cover other areas, social choice (Chichilnisky, *American Economic Review*, 1994, 427–434 and algebraic topology (Chichilnisky, *Bulletin of the American Mathematical Society*, 1993, 29, 189–207), rather than on the publication which contains my proofs on equilibrium. The comment's example is irrelevant to my results in Chichilnisky (*Economic Theory*, 1995, 5, 79–108) because it starts from different conditions. Limited arbitrage is always necessary and sufficient for the existence of a competitive equilibrium, with or without short sales, with the global cones as I defined them, and exactly as proved in Chichilnisky (*Economic Theory*, 1995, 5, 79–108).

JEL classification: C0; D5; G1

Keywords: Arbitrage; Market; Equilibrium; Global cones; Limited arbitrage

* UNESCO Professor of Mathematics and Economics and Director, Program on Information and Resources, Columbia University. E-mail: gc9@columbia.edu.

PII S0304-4068(97)00799-9

1. Introduction

Limited arbitrage is a unifying concept for resource allocation. Defined on the traders' endowments and preferences it is simultaneously necessary and sufficient for the existence of a competitive equilibrium, [1] for the non-emptiness of the core [2] and for the existence of satisfactory social choice rules. [3] Limited arbitrage extends to a topological invariant for competitive markets which contains exact information on the equilibrium, social choice and the core of all subeconomies, and predicts a failure of 'effective demand', (Chichilnisky, 1996b). In strictly regular economies, limited arbitrage is necessary and sufficient for the uniqueness of equilibrium (Chichilnisky, 1996c). For a complete presentation of my results on existence of an equilibrium the reader is referred to another paper in this issue (Chichilnisky, 1996b).

This paper responds to a recent comment (Monteiro et al., 1997) on my work on market equilibrium in Chichilnisky (1993a, 1994a, 1995a). I correct the comment's errors and show that its example is consistent with my results: limited arbitrage is always necessary and sufficient for the existence of a competitive equilibrium, with or without short sales, with the global cones which I introduced and exactly as proved in Chichilnisky (1995a). A summary of the response is as follows:

- The comment consists of a single example of a 'mixed economy', while none of my three publications which the comment addresses Chichilnisky (1993a, 1994a or 1995a) claim to cover mixed economies. The comment is therefore irrelevant to the publications on which it comments.
- The comment concentrates on my publications (Chichilnisky, 1993a, 1994a), which are brief announcements on equilibrium and cover other areas (algebraic topology and social choice theory). The comment therefore concentrates on the wrong references.
- Details and proofs of my results on equilibrium were published in Chichilnisky (1995a). These proofs are correct exactly as given. There is no inconsistency between the comment's example and any of my results, because they start from different conditions.
- My global cone G_h, introduced in Chichilnisky (1991, 1992, 1995a), is a well-defined concept which differs from the recession cones that are used elsewhere in the literature on no-arbitrage.
- Based on global cones I defined limited arbitrage, a concept which bounds trades and utility exactly as needed for the existence of a competitive equilib-

[1] See Chichilnisky (1992, 1993a, 1994a, b, 1995a, b, 1996a and Chichilnisky and Heal (1984, 1991a, 1992)).

[2] See Chichilnisky (1993, 1994b, 1995b, 1996a, 1997) and Chichilnisky and Heal (1992).

[3] See Chichilnisky (1991, 1994a, b, 1995b).

rium. The concepts of global cones and limited arbitrage are always the same throughout my work; the notation is adapted to the context.

- Limited arbitrage is necessary and sufficient for bounded and attainable gains from trade (Chichilnisky, 1995a).
- Limited arbitrage is necessary and sufficient for the compactness of the Pareto frontier. [4]
- All the above results in Chichilnisky (1995a, 1996a) hold in economies with or without short sales, and whether traders' indifferences contain half lines or not.

Furthermore

- Rather than a counterexample, the example given in the comment (Monteiro et al., 1997) is a special case of my results for mixed economies in Chichilnisky (1995b), reported in this issue (Chichilnisky, 1996b). The three authors cite Chichilnisky (1995b) but, for reasons which I leave the reader to surmise, fail to inform the reader of the consistency of their example and the results of my paper (Chichilnisky, 1995b).

How does limited arbitrage work? Limited arbitrage is strictly weaker than other conditions used to ensure the existence of an equilibrium, which is why it can be necessary and well as sufficient. Other conditions bound the feasible and individually rational trades. Defined by using global cones, limited arbitrage in unique in that it only bounds the utility levels which are achieved by such trades but not the trades themselves. The crucial insight is the equivalence of limited arbitrage and the compactness of the Pareto frontier in utility space (see footnote 4); this controls the existence of an equilibrium, of the core and of satisfactory social choice rules.

2. Mixed economies

In this section I correct two errors in the comment: one of misreading my conditions on preferences, and the second one is a misreading my definition of global cones. These errors invalidate the comment's claims.

(1) The comment (Monteiro et al., 1997) consists of one simple example of a 'mixed economy' with short trades, two traders and two goods. Mixed economies were not covered by my results. Trader one has a utility $u_1(x_1, x_2) = x_1 + x_2 + 2$

[4] The Pareto frontier is the set of undominated and individually rational utility values. The equivalence between the compactness of the Pareto frontier and limited arbitrage was pointed out first and established in a number of papers since 1984 (Chichilnisky and Heal, 1984, 1993; Chichilnisky, 1992, 1994b, 1995a, 1995b, 1996a) and was used there to prove existence of an equilibrium in markets with or without short sales, with finite or infinite dimensions, and whether traders' indifferences contain half line or not. Observe that the compactness of the Pareto frontier is irrelevant for Hart-type economies as studied by Page (1987) and others, because such economies are incomplete: their equilibria are typically inefficient and are not contained in the Pareto frontier.

$-\sqrt{(x_1-x_2)^2+4}$, and trader two's is $u_2(x_1, x_2)=x_1$.[5] Trader one's indifferences are bounded below, and the second trader's are not; trader two has indifferences with half lines and the other does not. This combination of preferences is a 'mixed' economy, a type of economy which I never claimed to cover in my work in (Chichilnisky, 1995a) which the comment addresses. My results in Chichilnisky (1995a) dealt only with homogeneous economies: the statement of my Theorem 1 in Chichilnisky (1995a) does not mention whether mixed economies were included, but it is clear to anyone who read my proofs that I only covered homogeneous economies as I considered the two cases quite separately – indifferences bounded below and those which are not bounded below – see p. 103, Section 7.0.1, lines 1–5.[6] Since the comment's example is *not* a homogeneous economy, it is not a counterexample to the results in the publications which it addresses, all of which pertain and refer to the proofs in Chichilnisky (1995a). Therefore the comment's example is irrelevant to the publications which it addresses.

(2) The three authors made another error, stating of their example "Thus limited arbitrage is satisfied".[7] However, the economy of their example does *not* satisfy limited arbitrage. The details are as follows. By my definition in Chichilnisky (1995a) the first trader in the example of (Monteiro et al., 1997, Section 3) has as global cone $G_1=\{x_1, x_2 : x_1 \geq 0 \text{ and } x_2 \geq 0\}$ because this trader's indifferences contain no half lines: i.e. case (b) of Chichilnisky (1995a, p. 85 (4)). However the three authors got this wrong, stating that my global cone G_6 is the open set they denote A_1 in the statement of their Theorem 3. My definition (p. 85, (4)) states clearly that when preferences contain no half lines (case (b)) the global cone G_h is the **closure** of the set A_1, i.e. it is the set they denote I_1 in Theorem 3 of Monteiro et al. (1997). The three authors confused I_1 with A_1. This error invalidates the statements in the comment.

The second trader in Monteiro et al. (1997) has half-lines in the indifferences (case (a) of Chichilnisky, 1995a, p. 85, (4)), and by my definition his/her global cone is the open half space $G_2=\{x_1, x_2 \in R^2 : x_1 > 0\}$, as the authors themselves point out in Theorem 3 of Section 3.2. It is now immediate to observe that, contrary to what the comment states, limited arbitrage is *not* satisfied in their

[5] See Monteiro et al. (1997, Section 3, Figures 1(a), 1(b), 2(a) and 2(b), and subsection 3.2).

[6] I quote from Chichilnisky (1995a, p. 103, Section 7.0.1, lines 1–5): "When all indifferences are bounded below the proof is identical to the case where $X=R^N_+$ a case where the result is known (cf. Arrow and Hahn (1)). Therefore we need only consider the case where $X=R^N$ and where the indifferences of the traders are not all bounded below". Observe that by Assumption 2 on p. 84 of Chichilnisky (1995a) if one indifference surface of a trader is bounded below, then all of his/her indifference surfaces are bounded below. Therefore the two exclusive cases considered these are: either all indifference surfaces of the traders bounded below, or none are. It is clear that the example in Section 3 of the comment Monteiro et al. (1997) does not satisfy my conditions because it mixes the two types of preferences.

[7] On the last line of Section 3.1 of Monteiro et al. (1997).

example in Section 3, Theorem 3 of Monteiro et al. (1997). The market cones are, respectively, $D_1 = \{(x_1, x_2): x_1 > 0 \text{ and } x_2 > 0\}$ and $D_2 = \{(x_1, x_2) = x_1 = 0\}$. Clearly $D_1 \cap D_2 = \emptyset$, so that limited arbitrage is not satisfied. It is not surprising, therefore, that their economy has no equilibrium, as the comment acknowledges. Limited arbitrage fails and no equilibrium exists. This is just a confirmation of my results. [8]

3. Markets without short sales

In this section I correct two other errors in the comment, about my results for the classic Arrow–Debreu economy without short sales, in Monteiro et al. (1997, Section 5, last paragraph). These errors invalidate the comment's claims.

(1) The three authors claim that my example in Fig. 4B of Chichilnisky (1995a, p. 88), has a competitive equilibrium but does not satisfy limited arbitrage. This is incorrect. This example does *not* have a competitive equilibrium. This example is in fact the classic example of non-existence of a competitive equilibrium with boundary endowments, first proposed by Kenneth Arrow and reported in his 1970 book with Frank Hahn. It has been known for over 25 years that the economy in this example has no competitive equilibrium. All this is in accordance with my Theorem 1 of Chichilnisky (1995a), and this confirms my results in Chichilnisky (1995a).

The details are as follows. Fig. 4B in Chichilnisky (1995a) is a two-person economy with trading space $X = R^2_+$. Trader one owns only the second good, $\Omega_1 = (0, x_2)$, [9] and trader two owns both goods, Ω_2 is in the interior of the positive orthant. [10] Trader one has a preference which is strictly increasing in both goods. [11] For trader one the global cone is $G_1(\Omega_1) = R^2_+ = \{(x_1, x_2): x_1 \geq 0 \text{ and } x_2 \geq 0\}$, i.e. the whole positive orthant [12] R^2_+ and by definition $D_1 = \{(x_1, x_2): x_1 > 0 \text{ and } x_2 > 0\}$. Trader two is indifferent in the second good as shown in Fig. 4B of Chichilnisky (1995a). Furthermore since this trader has an interior endowment [13] Ω_2, the only price in the set [14] $S(E)$ is the vector $v = (1, 0)$ as indicated in Fig. 4B. [15] Since $\langle v, \Omega_1 \rangle = 0$, by definition $S(E) \subset N$.

---

[8] Since there is no equilibrium and limited arbitrage fails, this example is clearly consistent with my results on mixed economies in Theorem 2 of Chichilnisky (1995b), a paper the authors acknowlegde they have, and they cite as Chichilnisky (1995a), see also this issue (Chichilnisky (1996b).

[9] See Chichilnisky (1995a, p. 89, lines 3 and 10–11).

[10] See lines 1–2 and 10–11, p. 89 of Chichilnisky (1995a).

[11] There is an obvious switch in the indices 1 and 2 here but in any case the argument is clear.

[12] As stated in lines 12–15 p. 89 of Chichilnisky (1995a).

[13] As stated in lines 1–2 and 10–11 p. 89 of Chichilnisky (1995a).

[14] The definition of $S(E)$ had a well known typographical error missing the expression "$\langle v, x_h - \Omega_h \rangle = 0$" in Chichilnisky (1995a), but this was corrected in the revised version of Chichilnisky (1995b) which the three authors acknowledge they have and cite as Chichilnisky (1995a).

[15] And as stated in lines 13–15 p. 89 of Chichilnisky (1995a).

Therefore by definition $D_1^+ = D_1 \cap S(E) = \emptyset$.[16] It follows that limited arbitrage is not satisfied in Fig. 4B, because $D_1^+ = \emptyset \Rightarrow D_1^+ \cap D_2^+ = \emptyset$. As Arrow pointed out a long time ago, this economy has no competitive equilibrium: the only possible supporting price is v, at which excess demand of trader one is not well defined. The comment went wrong by stating that in my example ''one trader likes only one good and is endowed with the other good''; see last paragraph of Section 5 of Monteiro et al. (1997). Neither of my two traders has the characteristic described by the comment: trader one only owns the second good and prefers both, and trader two owns both goods and prefers only the first. As detailed above, this example fails to have limited arbitrage since $D_1^+ \cap D_2^+ = \emptyset$. Since there is no equilibrium and limited arbitrage fails, this example confirms my Theorem 2 above (Chichilnisky, 1995b).

(2) Another example refers to Fig. 7 in p. 92 of Chichilnisky (1995a). This figure is the same as Fig. 4B above, and as stated in its legend, it has no competitive equilibrium. However the comment states that ''limited arbitrage is satisfied in this case (Figure 7)''; see last paragraph of Monteiro et al. (1997). However, as was shown in Example I above, the comment is wrong: Fig. 7 on p. 92 does not satisfy limited arbitrage. This is an example where there is no competitive equilibrium and limited arbitrage fails. This is consistent with my Theorem 1 in Chichilnisky (1995a).

4. Global cones

In this section I correct errors in the comment about my global cones. The commentators misread my definitions of global cones. Below I show that the global cone which I introduced is a well-defined concept which never changes. The notation is adapted to fit the context. On the basis of these global cones I defined limited arbitrage, and showed that it is exactly what is needed for the existence of an equilibrium, the core and social choice.

(1) In Section 5, under the title 'Impact of changes in Chichilnisky (1995a)', the comment states: ''Below we argue that global cones in Chichilnisky (1995a) are not well defined''. However the comment misread my definition of global cones and this error invalidates the statements in the comment.

My global cone G_h was introduced in 1991 (Chichilnisky, 1991, 1992, 1995a) is different from all other cones used in the no-arbitrage literature. This literature is based, instead, on the recession cones used by Rockafeller in 1970; see for example, Page (1987). In its more general form reported in Chichilnisky (1995a)

[16] As stated in page 89, line 13 and in my Fig. 4B of Chichilnisky (1995a).

my global cone G_h is the set of directions along which utility never ceases to increase. This is quite different from a recession cone, which is in this case the set of directions of non-decreasing utility. For example, for a constant function on R^N the recession cone is the whole of R^N, while my global cone is empty. [17] This shows that recession cones and global cones are quite different. A detailed exposition of my global cones is in Chichilnisky (1995a) which is the only publication among the three discussed in the comment to give details and proofs on limited arbitrage and equilibrium. Therefore I refer the reader to Chichilnisky (1995a, p. 84) where I define global cones. In Assumption 2, lines 3–5 of p. 84 I consider explicitly two cases: ''$\forall h$: (a) the directions of the gradients of each indifference surface which is not bounded below define a closed set, or (b) indifferences contain no half lines''. Preferences in case (a) always contain half lines in the indifferences, while those in case (b) never do. It is clear that whenever preferences have indifferences without half lines, we are in case (b). Because the two cases encompass different types of preferences, it can be expected that the same global cone will have slightly different realizations in the two cases and indeed they do. On p. 84 (4) I state: ''in case (a), the global cone is $A_h(\Omega_h)$ and in case (b), its closure $\bar{A}_h(\Omega_h)$''. In all cases, (a) and (b), my global cone is always the set of directions along which utility never ceases to increase. [18] This global cone is well defined and never changes. It appears that the commentators misread my papers.

(2) The comment mentions that cases (a) and (b) in Assumption 2 of Chichilnisky (1995a) may overlap, and that my global cones may therefore be ill-defined. [19] This is irrelevant. The only possible overlap is a trivial case: short sales do not matter because indifferences are bounded below *and* contain no half lines however in this case *any* definition of the global cone (A_h or $\bar{A}_h$) works, because all my conditions and results hold true trivially. In this case limited arbitrage is always trivially satisfied (with either cone A_h or $\bar{A}_h$), an equilibrium always exists, and the Pareto frontier is always compact because indifferences are bounded below. There is no ambiguity and no problem with my definition of global cones.

[17] The global cone is also different from the cone of directions where utility is strictly increasing: it is easy to show an increasing function on R^N_+ which is locally satiated at 0, in which the latter cone is empty while the global cone is the positive orthant.

[18] My global cone G_h is also identical to that I used in Chichilnisky (1995b, 1996a), in another paper in this issue (Chichilnisky, 1996b) and in Chichilnisky (1996c), papers which also contain details of the cones and proofs on equilibrium. In all cases my global cone is the same.

[19] In paragraph 1 of the subsection on 'impacts', Section 5.

5. Gains from trade

In this section I correct another error in the comment. The comment states that my concept of gains from trade is unrelated to equilibrium, when in fact they are very closely related.

(1) Monteiro et al. (1997) state: ''Chichilnisky's notion of gains from trade has no relevance for the existence of an equilibrium.'' [20] However, as shown in Proposition 2 page 90 of Chichilnisky (1995a) and mentioned in Proposition 1 of Chichilnisky (1994a, p. 428), [21] gains from trade are closely related to limited arbitrage, and therefore from Theorem 1 of Chichilnisky (1995a), closely related to the existence of a competitive equilibrium. Limited arbitrage is always necessary and sufficient for attainable and bounded gains from trade, in case (a) and (b). [22] As stated in Chichilnisky (1995a), the connection between limited arbitrage and bounded gains from trade is very close.

6. No half lines

In this section I correct another error in the comment, about my results for economies where preferences have no half lines. The comment implies that I do not cover this case, when in fact my work covers preferences without half lines in many publications, starting in 1984.

(1) Monteiro et al. (1997) state [23] that ''One of the main objectives of Chichilnisky (1994a, 1993a) appears to have been to obtain necessary and sufficient conditions for existence of equilibrium in terms of conditions limiting arbitrage for economic models in which agent's indifference surfaces are allowed to contain half lines''. This statement contains two errors. The main purpose Chichilnisky (1993a, 1994a) was *not* to prove a necessary and sufficient condition for the existence of an equilibrium: far from this, (1993a) contains only results in algebraic topology and (1994a) contains only results in social choice. [24] Of the three papers, the only one to contain details and proofs on equilibrium and limited arbitrage is Chichilnisky (1995a), in Theorem 1, p. 94, and this Theorem 1 is correct exactly as stated and proved.

(2) The second error in the statement quoted above is to imply that I focus on the case of indifferences with half lines (a). This is incorrect: although the case

[20] See Monteiro (1997, page 1, first paragraph).

[21] And in Chichilnisky (1996b).

[22] In case (a) of Chichilnisky (1995a) limited arbitrage is also necessary and sufficient for bounded gains from trade. This is also stated in Chichilnisky (1994a), and in Corollary 1 of Chichilnisky (1996b).

[23] In para. 5 of its Introduction.

[24] Both of these papers refer the reader to Chichilnisky (1995a) for details and proofs on equilibrium see Chichilnisky (1994a, p. 430, line 19) and Chichilnisky (1993a, p. 195, lines 8–9).

without half lines (b) is very simple I cover this case as well.[25] Indeed, my work on arbitrage and equilibrium in preferences *without half lines* goes back to Chichilnisky and Heal (1984, 1993): these two papers contain the first results on no-arbitrage, the compactness of the Pareto frontier and the existence of a competitive equilibrium in economies with or without short sales, with finite and infinite dimensions, and include preferences without half lines.[26]

Acknowledgements

Comments from Kenneth Arrow, Truman Bewley, Duncan Foley, Frank Hahn, Geoff Heal, Walt Heller, Ted Groves, Morris Hirsch, Paul Milgrom, Wayne Shafer and two anonymous referees are gratefully acknowledged.

References

Chichilnisky, G., 1991, Markets, arbitrage and social choice, presented at the conference, Columbia Celebrates Arrow's Contributions, Columbia University, New York, 27 October 1991; Working Paper No. 586 Columbia University, Department of Economics, December 1991, and CORE Discussion Paper No. 9342, CORE Universite Catolique de Louvain, Louvain la Neuve, Belgium, 1993.

Chichilnisky, G., 1992, Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, Working Paper No. 650, Columbia University, December.

Chichilnisky, G., 1993a, Intersecting families of sets and the topology of cones in economics, Bulletin of the American Mathematical Society 29, no. 2, 189–207.

Chichilnisky, G., 1993b, Limited arbitrage is necessary and sufficient for the existence of the core, Working Paper, Columbia University, revised 1994.

Chichilnisky, G., 1994a, Limited arbitrage, gains from trade and social diversity: A unified perspective on resource allocation, American Economic Review 84, no. 2, 427–434.

Chichilnisky, G., 1994b, Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium and the core and it limits voting cycles, Working Paper, Fall 1993, presented and distributed at the Yearly Meetings of the American Economic Association, Boston, 3–5 January 1994; Appeared as Limited arbitrage is necessary and sufficient for the nonemptiness of the Core, Economics Letters, 1996, 52, no. 2, 177–180.

Chichilnisky, G., 1995a, Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium with or without short sales', Economic Theory 5, no. 1, 79–108.

[25] Theorem 1 and Lemma 2 of Chichilnisky (1995a) cover preferences with half lines in their indifferences as well as those without, see p. 84 of Chichilnisky (1995a), Assumption 2, lines 3–4, where case (a) is with half lines and case (b) without half lines. See also the definition of global cones for cases (a) and (b) in p. 85, section 2.2 of Chichilnisky (1995a); Lemma 2 p. 96 and 103 of Chichilnisky (1995a) on the compactness of the Pareto frontier covers also cases (a) and (b), as does Theorem 1 on limited arbitrage being necessary and sufficient for the existence of a competitive equilibrium, p. 94 of Chichilnisky (1995a). See also Chichilniski (1996a) which cores strictly convex preferences.

[26] See, for example, Lemmas 4 and 5 and Theorem 1 of Chichilnisky and Heal (1993).

Chichilnisky, G., 1995b, A unified perspective on resource allocation: Limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium, the core, and social choice, CORE Discussion Paper No. 9527, Universite Catolique de Louvain, April 1995, revised November 1995 and March 1996. Invited for presentation at the International Economics Association Round Table on Social Choice, Vienna, May 1994, and for publication in: K. Arrow, A. Sen and T. Suzumura, eds., Social choice reexamined (Macmillan, 1997).

Chichilnisky, G., 1997, Market arbitrage, social choice and the core, Social Choice and Welfare 14, 161–198.

Cichilnisky, G. 1996a, Markets and games: A simple equivalence among the core, equilibrium and limited arbitrage, Metroeconomica 47, no. 3, 266–280.

Chichilnisky, G., 1996b, A topological invariant for competitive markets, Journal of Mathematical Economics, this issue.

Chichilnisky, G., 1996c, Limited arbitrage and uniqueness of equilibrium in strictly regular economies, Discussion Paper Series 9596–21, Department of Economics Columbia University, July.

Chichilnisky, G. and G.M. Heal, 1984, Existence of a competitive equilibrium in L_p and Sobolev spaces, IMA Preprint Series No. 79, Institute for Mathematics and its Applications, University of Minnesota, Minneapolis, Minnesota, June.

Chichilnisky, G. and G.M. Heal, 1991a, Arbitrage and the Pareto frontier, Working Paper, Columbia Business School.

Chichilnisky, G. and G.M. Heal, 1992, Arbitrage and equilibrium with infinitely many securities and commodities, Discussion Paper Series No. 618, Columbia University Department of Economics, July; Economic Theory, forthcoming.

Chichilnisky, G. and G.M. Heal, 1993, Existence of a competitive equilibrium in Sobolev spaces without bounds on short sales, Journal of Economic Theory 59, no. 2, 364–384.

Monteiro, P., F. Page and M. Wooders, 1997, Arbitrage, equilibrium and gains from trade: A counterexample, Journal of Mathematical Economics.

Page, F., 1987, Notes and comments to the editor: On equilibrium in Hart's securities exchange model, Journal of Economic Theory 41, 392–404.