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# A topological invariant for competitive markets

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**Abstract**

This paper summarizes the concepts of global cones and limited arbitrage introduced in Chichilnisky (*Economic Theory*, 1995, 5, 79–108), and the corresponding results establishing that limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium and for the compactness of Pareto frontier (announced in Chichilnisky (*American Economic Review*, 1992, 84, 427–434, and Chichilnisky (*Bulletin of the American Mathematical Society*, 1993, 29, 189–207). Using the same global cones I extend my earlier results to encompass 'mixed economies' based on Chichilnisky (CORE Discussion Paper No. 9527, 1995). I introduce a *topological invariant* for competitive markets which deepens the concept of limited arbitrage. This invariant encodes exact information on the equilibria and on the social diversity of the economy and all its subeconomies, and predicts a failure of effective demand.

*JEL classification:* D5; C0; G1

*Keywords:* Arbitrage; Topology; Markets; Social diversity; Limited arbitrage

## 1. Introduction

Limited arbitrage is central to resource allocation. It is simultaneously necessary and sufficient for the existence of a competitive equilibrium, [1] for the non-emptiness of the core [2] and for the existence of satisfactory social choice

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[1] See Chichilnisky (1991, 1992b, 1993c, 1994, 1995a, b and Chichilnsky and Heal (1992).

[2] See Chichilnisky (1993b, 1994, 1995b, 1996).


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rules,[3] recently, it was shown to be related to the uniqueness of market equilibrium (Chichilnisky, 1996c). Using the original concepts of global cones and limited arbitrage introduced in Chichilnisky (1992b, 1995a), I expand the scope of the theory by showing that limited arbitrage is necessary and sufficient for the existence of a competitive equilibrium in economies where different traders may have different types of preferences.[4]

What if limited arbitrage fails? To study this question I introduce a new concept: a topological invariant for competitive markets, denoted *CH*. This invariant defines various types of social diversity.[5] It deepens limited arbitrage, encoding exact information about the existence of equilibrium, the core and social choice for every subeconomy (Chichilnisky, 1995b). *CH* predicts a failure of 'effective demand' in Arrow–Debreu economies (Section 5). Based on my results on necessary and sufficient conditions for the nonempty intersection of families of sets in Chichilnisky (1993a), I show that the topological invariant *CH* gauges precisely which market cones intersect and which do not. A special case is when the invariant *CH* is zero: then all the market cones intersect and the economy has limited arbitrage.

Both limited arbitrage and the invariant *CH* are defined on the basis of a concept of global cone I introduced in 1991 (Chichilnisky, 1991, 1992b) and in Chichilnisky (1995a). My global cones, and the condition of limited arbitrage they imply, are always the same throughout my work including this paper; their notation is adapted to the context. Global cones consist of those directions along which the utility function does not achieve a maximum. They are generally smaller than the recession cones which are used in other literature on arbitrage and equilibrium: the latter cones consist of those directions along which the utility does not decrease.[6] This makes limited arbitrage unique in that it only bounds the feasible utility levels which the economy achieves. Other no-arbitrage conditions bound, instead, the set of individually rational and feasible allocations. Limited arbitrage is therefore weaker than the rest, which is why it can be simultaneously necessary and sufficient for the existence of equilibrium. It has been shown that limited arbitrage is equivalent to the compactness of the Pareto frontier in utility space,[7] and this controls the existence of equilibrium, the core and social choice rules, a crucial relationship which is extended here to mixed economies.

[3] See Chichilnisky (1991, 1992a, 1994, 1995b).

[4] Following Chichilnisky (1995b).

[5] See also Chichilnisky (1992a, 1995b).

[6] The difference is significant. When a utility function is constant, the recession cone is the whole space, while my global cone is empty. Recession cones are very different from global cones.

[7] The Pareto frontier is the set of undominated and individually rational utility values. This equivalence was established in Chichilnisky (1992b, 1994, 1995a, b, 1996a) and Chichilnisky and Heal (1984, 1991b, 1993), where it was used to prove the existence of equilibrium, the core and social choice rules.

*CH* is a topological invariant because it does not vary with continuous deformation of the commodity space. In particular it does not depend on the unit of measurements of the economy. Indeed, *CH* is only sensitive to the global properties of the preferences of the traders, as measured by their global cones.

This paper identifies an important role for social diversity. Markets need diversity to be useful, but as defined here too much diversity can be a hindrance. This raises questions about the role of diversity in resource allocation. Diversity is generally a positive force in the adaptation of a group to its environment. Is it possible that markets require less diversity to function that what would be ideal for society's successful adaptation? In more general terms: Are our economic institutions sustainable?

The paper is organized as follows. Sections 2–4 summarize my original results on the equivalence of limited arbitrage, the compactness of the Pareto frontier and the existence of an equilibrium given in Chichilnisky (1992b, 1994, 1995a) extending them on the basis of Chichilnisky (1995b) to encompass a larger class of mixed economies. Sections 5 and 6 cover social diversity and the failure of effective demand, and Section 7 introduces the topological invariant *CH* and develops its properties.

## 2. Definitions

This section addresses 'mixed economies' on the basis of Chichilnisky (1995b). This is a class of economies where different traders may have different types of preferences, and is larger than the class of economies I covered earlier in Chichilnisky (1995a). The definitions and results provided here are *identical* to those in Chichilnisky (1995a, 1996a) when restricted to the domain of 'homogeneous' economies covered by Chichilnisky (1995a, 1996a), but are correspondingly more general than those of Chichilnisky (1995a, 1996a) within mixed economies. [8] In all my work the global cones are the same and limited arbitrage is the same condition. The notation is adapted to the context.

[8] The results on limited arbitrage, the Pareto frontier and equilibrium in Chichilnisky (1995a, 1996a) hold in 'homogeneous' economies where either all indifferences have half lines or none do, and where either all traders have indifference surfaces bounded below, or none do; see p. 103, Section 7.0.1, lines 1–5 of Chichilnisky (1995a); mixed cases were covered in Chichilnisky (1995b). These results originated from a theorem in Chichilnisky and Heal (1984), a paper which was submitted for publication in February 1984 and was published in February 1993 (Chichilnisky and Heal, 1993). Chichilnisky and Heal (1993) introduced a non-arbitrage condition C which is identical to limited arbitrage for preferences without half lines, and proved that it is sufficient for the compactness of the Pareto frontier (Lemmas 4 without half lines, and proved that it is sufficient for the compactness of the Pareto frontier (Lemmas 4 and 5) and for the existence of an equilibrium (Theorem 1) with or without half lines, with or without short sales, and in finite or infinite dimensions. Chichilnisky (1995a) and Section 4 below discuss the literature further.

An Arrow–Debreu market $\mathbf{E} = \{X, \Omega_h, u_h, h = 1, \ldots, H\}$ has $H \geq 2$ traders, all with the same trading space $X$,[9] $X = R^N_+$ or $X = R^N$, $N \geq 2$. The results presented here also hold in infinite dimensions (Chichilnisky and Heal, 1992). Traders may have zero endowments of some goods. $\Omega_h \in R^N_+$ denotes trader $h$'s property rights. When $X = R^N_+$, $\Omega_h \neq 0$, $\forall h$, and $\Sigma_h \Omega_h = \Omega \gg 0$.

*Assumption 0.* Trader $h$ has a preference represented by a continuous, quasiconcave and increasing function $u_h : X \to R$.[10]

All the assumptions and all the results in this paper are ordinal,[11] therefore without loss of generality we choose a particular utility representation for the traders' preferences which satisfies: for all $h$, $u_h(0) = 0$ and $\sup_{\{x : x \in X\}} u_h = \infty$. This condition can be removed without changing any result at the cost of more notation.

*Assumption 1.* When $X = R^N_+$, $\forall r > 0$, any boundary ray $\Gamma$ is transversal to $u_h^{-1}(r)$.[12]

*Remark 1.* Assumption 1 means that either a boundary ray $\Gamma$ does not intersect $u_h^{-1}(r)$ or, if it does, the utility $u_h$ does not achieve a maximum on $\Gamma$.[13]

*Definition 1.* A preference on $R^N$ is *uniformly non-satiated* if it has a utility representation $u$ with a uniformly bounded rate of increase; e.g. when $u$ is smooth $\exists K$, $\varepsilon > 0$, such that $K > \| Du(x) \| > \varepsilon$, $\forall x, y \in R^N$.[14]

The following Assumption 2 restricts the rate of increase of the representing utility, and allows indifferences with or without 'flats'. Geometrically, this assumption includes two types of preferences: (a) those preferences where on every

[9] $R^N_+ = \{(x_1, \ldots, x_N) \in R^N : \forall i, x_i \geq 0\}$.

[10] This means that $u(x) \geq u(y)$ if $x \geq y$, and $u(x) > u(y)$ if $x \gg y$. Preferences need not be strictly increasing coordinatewise. If $x, y \in R^N$, $x \geq y \Leftrightarrow \forall i\, x_i \geq y_i$, $x > y \Leftrightarrow x \geq y$ and for some $i$, $x_i > y_i$, and $x \gg y \Leftrightarrow \forall i$, $x_i > y_i$.

[11] Namely independent of the utility representations.

[12] A boundary ray is an open half line starting from 0 contained in $\partial R^N_+$. Here we say that a manifold $X \subset R^N$ is *transversal* to another $Y$ at a point $x$ if either $X$ and $Y$ do not intersect, or when they do, neither of their respective tangent fields at $x$, $TX(x)$ and $TY(x)$, are contained in the other.

[13] Assumption 1 includes strictly convex preferences, preferences with indifferences of positive utility which do not intersect the boundary of the positive orthant such as Cobb Douglas and CES preferences, Leontief preferences $u(x, y) = \min(ax, by)$, preferences which are indifferent to one or more commodities, such as $u(x, y, z) = \sqrt{x + y}$, preferences with indifference surfaces which contain rays of $\partial R^N_+$ such as $u(x, y, z) = x$, and preferences defined on a neighborhood of the positive orthant or the whole space, and which are increasing along the boundaries, e.g. $u(x, y, z) = x + y + z$.

[14] Smoothness is not required but simplifies notation. In general one requires that $\forall x, y \in X\, \exists a, \varepsilon$ *and* $K > 0 : K \| x - y \| > | u(x) - u(y) |$ and $\sup_{(y : \| x - y \| < a)} | u(x) - u(y) | > \varepsilon \| x - y \|$.

indifference surface of a trader's utility the map $x \to Du(x)/\| Du(x) \|$ from allocations to normalized gradients is closed, and (b) those preferences where this map is open. The two types of preferences are quite different, indeed the two cases are exclusive: In case (a) every indifference surface contains half lines in every direction while in case (b) no indifference surface contains half lines in any direction. Case (b) was covered previously in Chichilnisky (1995a, 1996a). Formally:

*Assumption 2.* When $X = R^N$, preferences are uniformly non-satiated, and they satisfy one of the following two mutually exclusive conditions: (a) the normalized gradient to any closed set of indifferent vectors define a closed set [15] or (b) no indifference surface contains half lines.

*Remark 2.* There is one situation when the case $X = R^N$ is formally identical to the case where $X = R^N_+$ and where all traders have interior endowments: this is when $X = R^N$ and all traders' indifference surfaces are bounded below. [16] In this case short trades are allowed but no trader will trade below the utility level of the initial endowment so short trades are bounded below. It is worth observing that uniform non-satiation ensures that if one trader has one indifference surface bounded below, then all this trader's indifference surfaces are bounded below.

*Definition 2.* A *mixed economy* is one which has more than one type of preferences: some traders have preferences of type (a) and others of type (b), and/or some traders have indifferences bounded below, and others not.

*Remark 3.* Sections 2–7 of this paper and Chichilnisky (1995b) cover mixed economies; in Chichilnisky (1995a) either all traders had indifferences bounded below or none did, and either all traders were of type (a), or all traders were of type (b); see Chichilnisky (1995a, p. 103, Section 7.0.1., lines 1–5). The definitions and results are however identical to those in Chichilnisky (1995a) for the cases which that paper covered.

### 2.1. Global and market cones

This section defines global cones, a concept which hs the same throughout my work and appeared first in its most general form in Chichilnisky (1995a). The

[15] I.e. the map $x \to Du_h(x)/\| Du_h$ (x $\|$ maps closed sets of an indifference surface into closed sets in the unit sphere. This means that the gradient directions to any indifference surface define a closed set, and corresponds to case (a) in Chichilnisky (1995a), encompassing here preferences whose indifferences are bounded below or not.

[16] A set $S$ is said to be *bounded below* when $\exists y \in R^N : \forall x \in S,\ x \geq y$.

cones presented here are identical to those in Chichilnisky (1995a), for the homogeneous economies considered in that paper. The notation is adapted to fit the context. Two cases, $X = R^N$ and $X = R^N_+$, are considered separately.

● Consider first $X = R^N$.

*Definition 3.* For trader $h$ define the cone of directions $A_h(\Omega_h)$ along which utility increases without bound:

$$A_h(\Omega_h) = \{x \in X : \forall y \in X, \exists \lambda > 0 : u_h(\Omega_h + \lambda x) > u_h(y)\}.$$

The rays of this cone intersect all indifference surfaces preferred to $\Omega_h$. In case (a) the cone $A_h(\Omega_h)$ is open (Proposition 1 of Chichilnisky (1995a)). When augmented by the part of its boundary along which utility never ceases to increase $A_h(\Omega_h)$ defines the *global cone* $G_h(\Omega_h)$–both cones are new in the literature: [17]

$$G_h(\Omega_h) = \left\{ x \in X \text{ and } \sim \exists \max_{\mu \geq 0} u_h(\Omega_h + \mu x) \right\}. \quad (1)$$

It is important to observe that this cone is identical to the global cone $G_h(\Omega_h)$ which was used in Chichilnisky (1995a) to define limited arbitrage; see Chichilnisky (1995a, p. 85, (4)). The alternative notation used here is simplifying in that the cone $G_h(\Omega_h)$ treats all convex preferences in a unified way. In all cases under Assumption 2, $G_h(\Omega_h)$ is shown in Proposition 3 of the Appendix to be identical to the global cone in Chichilnisky (1995a), namely in case (a) when preferences have half lines $G_h(\Omega_h)$ equals $A_h(\Omega_h)$, and it equals its closure when indifferences contain no half lines, i.e. in case (b); $G_h(\Omega_h)$ is independent of the initial endowment $\Omega_h$. [18]

The *market cone* of trader $h$ is

$$D_h(\Omega_h) = \{z \in X : \forall y \in G_h(\Omega_h), \langle x, y \rangle > 0.\} \quad (2)$$

$D_h$ is the convex cone of prices assigning strictly positive value to all directions in $G_h(\Omega_h)$.

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[17] The cone $A_h(\Omega_h)$ has points in common with Debreu's (1959) 'asymptotic cone' corresponding to the preferred set of $u_h$ at the initial endowment $\Omega_h$, in that along any of the rays of $A_h(\Omega_h)$ utility increases. Under Assumption 1, its closure $\bar{A}(\Omega_h)$, equals the 'recession' cone used by Rockafeller in the 1960s, *but* not generally: when preferences have 'fans' as defined in Chichilnisky (1995b), then the recession cone differs from $A_h(\Omega_h)$. Along the rays in $A_h(\Omega_h)$ not only does utility increase forever, but it increases beyond the utility level of any other vector in the space. This condition need not be satisfied by asymptotic cones or by recession cones. For example, for Leontief preferences the recession cone through the endowment in the closure of the upper contour, which includes the indifference curve itself. By contrast, the cone $A_h(\Omega_h)$ is the interior of the upper contour set. Related concepts appeared in Chichilnisky (1976, 1986, 1991, 1995a, 1996b); otherwise there is no precedent in the literature for global cones $A_h(\Omega_h)$ or $\bar{A}_h(\Omega_h)$.

[18] The cone $G_h(\Omega_h)$ is identical to that defined in Chichilnisky (1995a, p. 18 (4)); because it is the same at every allocation under Assumption 2, it is called 'uniform' and denoted also $G_h$.

- Consider next the case: $X = R^N_+$.

The *market cone of trader h* is:

$$D^+_h(\Omega_h) = D_h(\Omega_h) \cap S(\mathbf{E}), \quad \text{if } S(\mathbf{E}) \subset \mathbf{N},$$
$$= D_h(\Omega_h), \quad \text{otherwise}. \tag{3}$$

where $S(\mathbf{E})$ is the set of *supports to rational affordable efficient* allocations, defined as [19]

$$S(\mathbf{E}) = \{v \in R^N : \text{if } (x_1 \dots x_H) \in \Gamma, \langle v, x_h - \Omega_h \rangle = 0,$$
$$\text{and } u_h(z_h) \geq u_h(x_h) \text{ then } \langle v, z_h - x_h \rangle \geq 0, \forall h\}, \tag{4}$$

and where $\mathbf{N}$ is the set of prices orthogonal to the endowments, defined as [20]

$$\mathbf{N} = \{v \in R^N_+ - \{0\} : \exists h \text{ with } \langle v, \Omega_h \rangle = 0\}. \tag{5}$$

The market cones $D^+_h(\Omega_h)$ typically vary with the initial endowments. When $\forall h \Omega_h \in R^N_{++}$, $D^+_h(\Omega_h) = D_h(\Omega_h)$.[21]

### 2.2. Limited arbitrage

This section defines my concept of limited arbitrage, the same concept that I have used throughout my work. Limited arbitrage was defined in Chichilnisky (1995a) is based on the global cones introduced there and defined above.

*Definition 4.* If $X = R^N$, then $\mathbf{E}$ satisfies *limited arbitrage* when

$$(LA) \bigcap_{h=1}^{H} D_h \neq \emptyset.$$

When global cones are independent of the initial endowments, this condition is satisfied simultaneously at every set of endowments. When preferences are in case (b) $(LA)$ is identical to the *no-arbitrage condition C* introduced in Chichilnisky and Heal (1984, p. 374), and used there to prove the compactness of the Pareto frontier and the existence of an equilibrium with short sales. This definition of limited arbitrage is also identical to that in Chichilnisky (1995a, p. 89, Section 3.1) in the cases which that paper covers.

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[19] $\Gamma = \{(x_1, \dots, x_H) : \forall, u_h(x_h) \geqslant u_h(\Omega_h)\}$. *The expression* '$(v, x_h - \Omega_h) = 0$' was missing in the definition of $S(E)$ in Chichilnisky (1995a) due to a typographical error, which was corrected in the revised version of Chichilnisky (1995b).

[20] $N$ is empty when $\forall h$, $\Omega_h \gg 0$.

[21] $R^N_{++}$ is the interior of $R^N_+$. The market cone $D^+_h$ contains $R^N_{++}$ when $S(E)$ has a vector assigning strictly positive income to all individuals. If some trader has zero income, then this trader must have a boundary endowment.

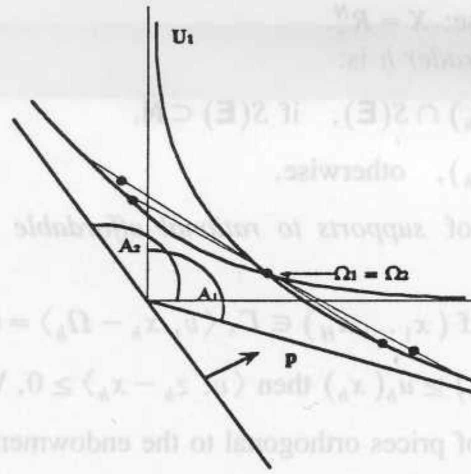


Fig. 1. Limited arbitrage is satisfied. The two global cones lie in the halfspace defined by $P$. There are no feasible trades that increase utilities without limit: these would consist of pairs of points symmetrically placed about the common initial endowment. and such pairs of points lead to utility values below these of the endowments at a bounded distance from the initial endowments.

*Definition 5.* If $X = R_+^N$, then **E** satisfies *limited arbitrage* when

$$(LA^+) \bigcap_{h=1}^{H} D_h^+(\Omega_h) \neq \emptyset. \tag{6}$$

This definition is identical to that in Chichilnisky (1995a, (8), p. 91, Section 3.3).

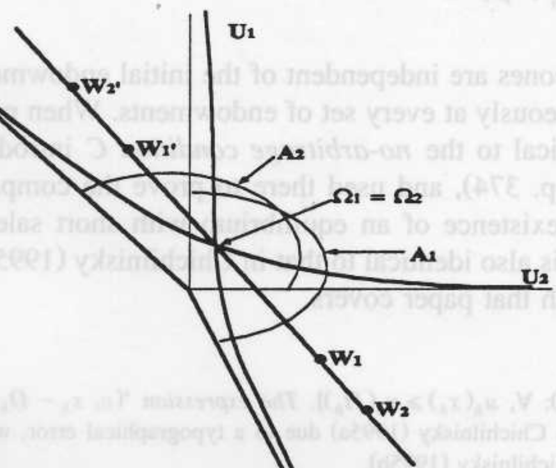


Fig. 2. Limited arbitrage does not hold. The global cones are not contained in a half space, and there are sequence of feasible allocations such as $(W_1, W_1', W_2, W_2')$ which produce unbounded utilities.

The set of *feasible individually rational trades* is $\Upsilon = \{(x_1, \ldots, x_H) \in X^H : \forall h,\ u_h(x_h) \geq u_h(\Omega_h)$, and $\sum_{h=1}^{H} x_h \leq \Omega\}$. The set of feasible individually rational *utility allocations* is

$$U(\mathbf{E}) = \{U = (U_1, \ldots, U_H) \in R^H : U_h = u_h(x_h) \geq u_h(\Omega_h) \text{ where } (x_1, \ldots, x_H) \in \Upsilon\}.$$

The *Pareto frontier* $P(\mathbf{E})$ is the set of undominated vectors in $U(\mathbf{E})$:

$$P(\mathbf{E}) = \{U \in U(\mathbf{E}) : \sim \exists V \in U(\mathbf{E}) \text{ with } V > U\}.$$

A *competitive equilibrium* is a price vector $p^* \in R_+^N$ and an allocation $(x_1^* \ldots x_H^*) \in X^H$ such that $\sum_{h=1}^{H} x_h^* - \Omega_h = 0$ and $x_h^*$ maximizes $u_h$ over the budget set $B_h(p^*) = \{x \in X : \langle x - \Omega_h, p^* \rangle = 0\}$.

### 2.3. A financial interpretation of limited arbitrage

It is useful to show the connection between limited arbitrage and the notion of 'no-arbitrage' used in finance. The concepts are generally different, but in certain cases they coincide. In financial markets an arbitrage opportunity exists when gains can be made at no cost or, equivalently, by taking no risks. The simplest illustration of the link between limited arbitrage and no-arbitrage is an economy $\mathbf{E}$ where the traders' initial endowments are zero, $\Omega_h = 0$, and the normalized gradient of a closed set of indifference vectors define a closed set, case (a). Here *no-arbitrage* at the initial endowments means that there are no trades which could increase the traders' utilities at zero cost: gains from trade must be zero. By contrast, *limited arbitrage* means that no trader can increase utility beyond a given bound at zero cost; as seen below in Subsection 3.1, gains from trade are bounded. In brief: in this example *no-arbitrage* requires that there should be no gains from trade at zero cost while *limited arbitrage* requires that there should be only bounded or limited gains from trade at zero cost.

In case (b) limited arbitrage has a stricter meaning: it implies that traders will not engage in unboundedly large transactions; this is identical to the no-arbitrage condition C introduced in Chichilnisky and Heal (1984, p. 374).

When the traders' utilities are linear functions, the two concepts coincide: there is limited arbitrage if and only if there is no-arbitrage as defined in finance. In brief: in linear economies, limited arbitrage 'collapses' into no-arbitrage.

## 3. Limited arbitrage and the compactness of the Pareto frontier

The Pareto frontier $P(\mathbf{E})$ is the set of feasible, individually rational and undominated utility allocations. The compactness (and non-emptiness) of the Pareto frontier is a crucial step in establishing the existence of a competitive equilibrium.

It is useful to clarify the difference between limited arbitrage and conditions used elsewhere.[22] Other conditions used in the literature require a compact set of feasible individually rational allocations $\Upsilon$.[23] For example, the no-arbitrage condition C of Chichilnisky and Heal (1993) ensures a compact set of individually rational and feasible trades $\Upsilon$, and thus the compactness of $P(\mathbf{E})$.[24] By contrast, limited arbitrage ensures a compact utility possibility set, *without requiring* the compactness of the underlying set of trades $\Upsilon$; see Chichilnisky (1991, 1992a, b, 1994, 1995a, b, 1996b) and Chichilnisky and Heal (1991b). It has been established within homogeneous economies that limited arbitrage is necessary and sufficient for a compact utility possibility set $U(\mathbf{E})$, and hence the compactness of its boundary $P(\mathbf{E})$;[25] for details see Chichilnisky (1992b, 1995a) and Chichilnisky and Heal (1991a, 1993). The following theorem extends this result to mixed economies following Chichilnisky (1995b).

*Theorem 1. Consider an economy* $\mathbf{E}$ *as defined in Section* 1. *Then limited arbitrage is necessary and sufficient for the compactness of the set of feasible and individually rational utility allocations* $U(\mathbf{E})$, *and therefore necessary and suffi-*

[22] Case (b), when indifferences contain no half lines, is particularly simple: under limited arbitrage feasible and rational trades always define a compact set, exactly as in economies without short sales, Chichilnisky (1996a).

[23] Without short sales the set $\Upsilon$ is always compact. In economies with short sales,Chichilnisky and Heal (1984, 1993), introduced the 'no-arbitrage condition C', and similar conditions were used in Werner (1987), Nielsen (1989), Koutsougeras (1995) (among others), the latter three based on recession cones. Any no-arbitrage condition based on recession cones amounts to a compact set of feasible and desirable allocations $\Upsilon$ in commodity space, see e.g. Lemma 2 of Chichilnisky (1995a, 1996a section 4.2) and Lemma 10.1, Section 10, p. 20 of Koutsougeras (1995). Werner's sufficient conditions for existence is somewhat weaker because he removes some constant directions from the recession cone, but like all the others it is strictly stronger than limited arbitrage. He requires $S = \cap S_i \neq \emptyset$ (see Theorem 1 on p. 1410 of Werner, 1987), where $S_i = \{p : \langle p, x \rangle > 0\}$ for all $x \in W_i\}$ (bottom of p. 1409), and where $W_i$ is the recession cone minus the directions along which the utility is constant, see 1408, p. 6, last two lines. Observe that $W_i$ contains many directions which are not included in $G_h$ such as those directions along which the preference is eventually constant without being constant. Such directions are not part of the global cone $G_h$ defined here and in Chichilnisky (1995a). Therefore the cone $W_i$ is strictly larger than the global cone $G_h$, and the condition $S \neq \emptyset$ is strictly stronger than limited arbitrage as defined here and in Chichilnisky (1995a). Werner's condition is therefore not necessary for existence of an equilibrium unless indifferences contain no half lines, see also Proposition 2, (ii), p. 1410 of Werner (1987). In this latter case, Werner's condition bounds the set of all feasible individually rational allocations $\Upsilon$ as do all other non-arbitrage conditions except for limited arbitrage.

[24] Lemmas 4 and 5 of Chichilnisky and Heal (1984) prove the compactness of $P(\mathbf{E})$ using its 'no-arbitrage condition C' which is identical to limited arbitrage when indifferences have no half lines; see Chichilnisky and Heal (1984) and Chichilnisky (1995a, b).

[25] Lemma 2 of Chichilnisky (1995a) established that limited arbitrage is sufficient for compactness of $P(\mathbf{E})$ and Theorem 1 of Chichilnisky (1995a), by the second welfare theorem, proved that limited arbitrage is necessary for compactness of $P(\mathbf{E})$, because it is necessary for the existence of an equilibrium.

*cient for the compactness and non-emptiness of the Pareto frontier $P(\mathbf{E})$. Feasible allocations and undominated utility allocations may however define unbounded sets.*

*Proof.* See the appendix. □

*Proposition 1. When $X = R^N$, limited arbitrage implies that the Pareto frontier $P(\mathbf{E}) \subset R^N_+$ is homeomorphic to a simplex.* [26]

*Proof.* This follows from Theorem 1 and the convexity of preferences, cf. Arrow and Hahn (1971). □

### 3.1. Interpretation of limited arbitrage as bounded gains from trade when $X = R^N$

Limited arbitrage has a simple interpretation in terms of gains from trade when $X = R^N$. Gains from trade are defined as follows:

$$\mathbf{G}(\mathbf{E}) = \sup_{(x_1,\ldots,x_H) \in \Upsilon} \sum_{h=1}^{H} u_h(x_h) - u_h(\Omega_h);$$

$\Upsilon$ was defined above as the set of feasible and individually rational allocations.

*Proposition 2. The economy $\mathbf{E}$ satisfies limited arbitrage if and only if it has bounded gains from trade which are attainable, i.e. $\exists x^* \in \Upsilon$:*

$$\mathbf{G}(\mathbf{E}) = \sum_{h=1}^{H} u_h(x_h^*) - u_h(\Omega_h) < \infty.$$

*Proof.* Proposition 3 in the Appendix shows that under the conditions global cones are independent of the initial endowments, so we may assume without loss of generality that $\Omega_h = 0$ for all $h$. Sufficiency first. The sum $\sum_{h=1}^{H} u_h$ defines a continuous function of $(u_1,\ldots,u_H) \in R^H_+$; this continuous function always attains a maximum when the vector $u_1,\ldots,u_h$ varies over the set $U(\mathbf{E}) \subset R^H_+$, which is compact by Theorem 1 when limited arbitrage is satisfied. The converse is immediate from the proof of necessity in Theorem 1 (see the Appendix), which establishes that when limited arbitrage fails, there exist no undominated individually rational utility allocations, i.e. $P(\mathbf{E})$ is empty. □

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[26] A topological space $X$ is homeomorphic to another $Y$ when there exists a on-to-one onto map $f: X \to Y$ which is continuous and has a continuous inverse.

[27] In the following equation '$\infty$' must be replaced by $\sup_{(x: x \in X)} (\sum_{h=1}^{H} u_h(x) - u_h(\Omega_h))$ when $\forall h, \sup_{x \in X} u_h(x) < \infty$. The difference between Proposition 2 and Corollary 1 is that bounded gains from trade need not be attainable.

*Corollary 1. In case* (a) *the economy has limited arbitrage if and only if it has bounded gains from trade, i.e.* [27]

$$\mathbf{G}(\mathbf{E}) < \infty.$$

*Proof.* Recall that in case (a), $G_h = A_h$ by Proposition 3 in the Appendix. In this case limited arbitrage is also necessary for bounded gains from trade. □

## 4. Competitive equilibrium and limited arbitrage

It was proved in Chichilnisky (1995a) that limited arbitrage is simultaneously necessary and sufficient for existence of a competitive equilibrium in economies with or without short sales. [28] This result was established in Chichilnisky (1995a) for economies where preferences were homogeneous: either all preferences are of type (a) or all of type (b), and either all have indifferences bounded below or none do. Below I show that this is true for mixed economies, where some preferences may be of type (a) and others of type (b), where some indifferences are bounded below and others are not. This follows Chichilnisky (1995b). This result includes the classic economy of Arrow and Debreu without short sales, which was neglected previously in the literature on no-arbitrage. The equivalence between limited arbitrage and equilibrium extends to economies with infinitely many markets, see Chichilnisky and Heal (1992).

Recall that limited arbitrage implies a compact set of feasible and individually rational utility allocations $U(\mathbf{E})$, *without requiring* that the set of underlying trades $\mathcal{T}$ be compact. Therefore under limited arbitrage the set of equilibria, the set of undominated utility allocations and the core may all be unbounded when $X = R^N$.

[28] Chichilnisky and Heal (1984), Hart (1974) Hammond (1983) and Werner (1987) among others, have defined various no-arbitrage conditions which they prove, under certain conditions on preferences, to be sufficient for existence of equilibrium. Hart and Hammond study asset market models which are incomplete economies because they lack forward markets, and therefore have typically inefficient equilibria. None of these no-arbitrage conditions is generally necessary for existence. For the special case (b), within economies with short sales (which exclude the classic Arrow–Debreu's market), and where recession cones are assumed to be uniform, Werner (1987) remarks correctly (p. 1410, last para.) that another related condition (p. 1410, line -3) is necessary for existence, without however providing a complete proof of the equivalence between the condition which is necessary and that which is sufficient. The two conditions in Werner (1987) are different and are defined on different sets of cones: the sufficient condition is defined on cones $S_i$ (p. 1410, line -14) while the necessary condition is defined on other cones, $D_4$ (p. 1410, -3). The equivalence between the two cones depends on properties of yet another family of cones $W_i$ (see p. 1410, lines 13-4). The definition of $W_i$ on page 1408, line -15 shows that $W_i$ is different from the recession cone $R_i$, (which are uniform by assumption) and therefore the cone $W_i$ need not be uniform even when the recession cones are, as needed in Werner's Proposition 2.

*Theorem 2. Consider an economy* $\mathbf{E} = \{X, u_h, \Omega_h, h = 1, \ldots, H\}$, *where* $H \geq 2$, *with* $X = R^N$ *or* $X = R^N_+$ *and* $N \geq 1$. *Then the following two properties are equivalent*:

(i) *The economy* $\mathbf{E}$ *has limited arbitrage.*
(ii) *The economy* $\mathbf{E}$ *has a competitive equilibrium.*

*Proof.* See the appendix. □

*Remark 5.* When $X = R^N$ the set of competitive equilibria need not be bounded.

### 4.1. Subeconomies with competitive equilibria

As shown in Chichilnisky (1995a) the condition of limited arbitrage need not be tested on all traders simultaneously: in the case of $R^N$, it needs only be satisfied on subeconomies with no more traders than the number of commodities in the economy, $N$, plus one.

*Definition 6.* A $k$-trader sub-economy of $\mathbf{E}$ is an economy $\mathbf{F}$ consisting of a subset of $k \leq H$ traders in $\mathbf{E}$, each with the endowments and preferences as in $\mathbf{E}$: $\mathbf{F} = \{X, u_h, \Omega_h, h \in J \subset \{1, \ldots, H\}$, cardinality $(J) = k \leq H\}$.

*Theorem 3. The following four properties of an economy* $\mathbf{E}$ *with trading space* $R^N$ *are equivalent*:

(i) $\mathbf{E}$ *has a competitive equilibrium.*
(ii) *Every sub-economy of* $\mathbf{E}$ *with at most* $N+1$ *traders has a competitive equilibrium.*
(iii) $\mathbf{E}$ *has limited arbitrage.*
(iv) $\mathbf{E}$ *has limited arbitrage for any subset of traders with no more that* $N+1$ *members.*

*Proof.* Theorem 2 implies (i) ⇔ (iii) and (ii) ⇔ (iv). That (iii) ⇔ (iv) follows from Helly's theorem. which is a corollary in Chichilnisky (1993a): Consider a family $\{U_i\}_{i=1 \ldots H}$ of convex sets in $R^N$, $H, N \geq 1$. Then

$$\bigcap_{i=1}^{H} U_i \neq \emptyset \text{ if and only if } \bigcap_{j \in J} U_i \neq \overline{\emptyset},$$

for any subset of indices $J \subset \{1 \ldots H\}$ having at most $N+1$ elements. In particular, an economy $\mathbf{E}$ satisfies limited arbitrage, if and only if it satisfies limited arbitrage for any subset of $k = N+1$ traders, where $N$ is the number of commodities in the economy $\mathbf{E}$. □

## 5. A failure of effective demand when $\mathbf{X} = \mathbf{R}^N_+$

In economies without short sales $X = R^N_+$ a *compensated equilibrium* or *quasiequilibrium* always exists, Arrow and Hahn (1971) and Negishi (1960).

However, when limited arbitrage fails, quasiequilibrium are ill-behaved; I show below that every quasiequilibrium has a failure of 'effective' demand, in the sense of Arrow and Hahn (1971, p. 345).

*Definition 7.* A compensated equilibrium or quasiequilibrium is a price $p^*$ and an allocation $x^*$ at which every trader minimizes the cost of achieving the given utility level, and $\Sigma_{h=1}^{H}(x_h^* - \Omega_h) = 0$.

Although a quasiequilibrium allocation $x^*$ satisfies $\Sigma_{h=1}^{H}(x_h^* - \Omega_h) = 0$, $x^*$ can exhibit 'excess demand', if demand is computed from utility maximization under a budget constraint. This is because at the quasiequilibrium allocation traders minimize costs, but may not maximize utility. If at a quasiequilibrium allocation every trader maximizes utility, then the quasiequilibrium is also a competitive equilibrium.

*Definition 8.* An allocation and a price exhibit a failure of effective demand in the sense of Arrow and Hahn (1971) when in some markets there exists excess demand (computed from utility maximization subject to a budget constraint), but the value of this excess demand is zero.

*Theorem 4. When $X = R_+^N$ a failure of limited arbitrage implies that there is a failure of effective demand: every quasiequilibrium has excess demand, but the market value of this demand is zero.*

*Proof.* A quasiequilibrium price $p^* = (p_1^*, \ldots, p_N^*) \in R_+^N$ always exists because $X = R_+^N$. As seen in Theorem 2 if limited arbitrage fails there is no competitive equilibrium. This implies that at any quasiequilibrium some trader has zero income (for otherwise a quasiequilibrium is always a competitive equilibrium). Therefore for at least one market $i$, $p_i^* = 0$, and the value of excess demand in the $i$th market is therefore zero. For some trader $h$, $p^* \notin D_h^+$ (since otherwise the quasiequilibrium would be a competitive equilibrium) so that there is no maximum for $u_h$ on the budget set defined by $p^*$. Therefore there is excess demand at $p^*$ in the $i$th market, but the value of excess demand in this market is zero. □

## 6. Social diversity and limited arbitrage

If the economy does not have limited arbitrage, it is called *socially diverse*: *Definition 9.* When $X = R^N$, the economy is socially diverse if $\bigcap_{h=1}^{H} D_h = \emptyset$.

This concept is independent of the units of measurement or choice of numeraire. Social diversity admits as many different 'shades' as traders:

*Definition 10.* The economy $\mathbf{E}$ has *index of diversity* $I(\mathbf{E}) = H - K$ if $K+1$ is the smallest number such that $\exists J \subset \{1 \ldots H\}$ with cardinality of $J = K+1$, and

$\bigcap_{h \in J} D_h = \emptyset$. The index $I(\mathbf{E})$ ranges between 0 and $H-1$: the larger the index, the larger the social diversity. The index is smallest $(=0)$ when all the market cones intersect: then all social diversity disappears, and the economy has limited arbitrage.

*Theorem 5. The index of social diversity is $I(E)$ if and only if $H - I(E)$ is the maximum number of traders for which every subeconomy has a competitive equilibrium.*

*Proof.* This is immediate from Theorem 3. □

## 7. A topological invariant for competitive markets

This section introduces a new concept, a *topological invariant for competitive markets*. This is an algebraic object which describes important properties of the market – its social diversity and the existence of an equilibrium for it or for its subeconomies – in a way that is robust, namely independent of the units of measurement. The topological invariant $CH$ contains exact information about the resource allocation properties of the economy, as shown below. Furthermore it is computable by simple algorithms from the initial data of the problem: its endowments and preferences.

Important properties of an economy $\mathbf{E}$ with short sales can be described in terms of the properties of a family of cohomology rings denoted $CH(\mathbf{E})$. The cohomology rings [29] of a space $Y$ contain information about the topological structure of $Y$, namely those properties of the space which remain invariant when the space is deformed as if it was made of rubber. For formal definitions of the algebraic topology concepts used here a standard textbook is Spanier (1979).

The following concept, called a **nerve**, defines a combinatorial object, called a **simplicial complex**, from any family of sets. Starting from any arbitrary family of sets in $R^N$, one defines from it a triangulated set, obtained by 'pasting up' simplices in a well ordered fashion (see Spanier, 1979) called a *simplicial complex*. The simplicial complex created from the family of sets is usually called the 'nerve of the family of sets'. The procedure for transforming any family of sets into a simplicial complex is as follows: each set is represented by a point, which is a vertex in the triangulated space; the intersection of any two sets in the family is represented as a one-simplex, the intersection of any three sets as a 2-simplex, etc. It is simple to see that putting all this together one obtains a simplicial complex (Spanier, 1979). Formally:

[29] A ring is a set $Q$ endowed with two operations, denoted $+$ and $\times$; the operation $+$ must define a group structure for $Q$ (every element has an inverse under $+$) and the operation $\times$ defines a semi group structure for $Q$; both operations together satisfy a distributive relation. A typical example of a ring is the set of the integers, another is the rational numbers, both with addition and multiplication.

*Definition 11.* The *nerve* of a family of subsets $\{V_i\}_{i=1,\dots,L}$ in $R^M$, denoted

$$\text{nerve}\{V_i\}_{i=1,\dots,L}$$

is a simplicial complex defined as follows: each subfamily of $k+1$ sets in $\{V_i\}_{i=1,\dots,L}$ with non-empty intersection is a $k$-simplex of the nerve $\{V_i\}_{i=1,\dots,L}$.

Now that the nerve of a family of sets is defined, one looks at the global topology of this simplicial complex, in words, how many 'holes' it has, and of what type. This is measured by the cohomology rings of the space, which are groups with an additional operation, which measure precisely the number and types of holes which the space has. The cohomology rings are computable by standard algorithms once the market cones of the economy are known. See Spanier (1979) for definitions. Now we are ready to define our topological invariant.

A *subfamily* of the family of sets $\{D_h\}_{h=1,\dots,H}$ is a family consisting of some of the sets in $\{D_h\}_{h=1,\dots,H}$, and is indicated $\{D_h\}_{h\in Q}$, where $Q \subset \{1,\dots,H\}$. The *topological invariant* $CH(\mathbf{E})$ of the economy $\mathbf{E}$ with $X = R^N$ is the family of reduced cohomology rings [30] of the simplicial complexes defined by all subfamilies $\{D_h\}_{h\in Q}$ of the family $\{D_h\}_{h=1,2,\dots,H}$, i.e. the cohomology rings of $nerve\{D_h\}_{h\in Q}$ for every $Q \subset \{1,\dots,H\}$:

$$CH(\mathbf{E}) = \left\{H^*\left(\text{nerve}\{D_h\}_{h\in Q}, \forall Q \subset \{1,\dots,H\}\right\}\right.$$

In the following result I consider continuous deformations of the economy which preserve all the Assumptions in Section 2. Any such deformation will preserve our topological invariant. The following result shows how much information is encoded in $CH(\mathbf{E})$. It shows that the existence of an equilibrium for the economy and its subeconomies are purely topological properties, and they are predicted exactly by the number and the type of 'holes' in the nerve of the family of market cones of the economy, i.e. by the cohomology rings of this nerve. This result depends on a new theorem in Chichilnisky (1993a) which established that a family of finitely many convex sets has non empty intersection if and only if each subfamily has a contractible union:

*Theorem 6. The economy* $\mathbf{E}$ *with H traders has limited arbitrage, and therefore a competitive equilibrium if and only if:*

$$CH(\mathbf{E}) = 0,$$

$$\text{i.e. } \forall Q \subset \{1,\dots,H\},\ \mathbf{H}^*\left(\text{nerve}\{D_h\}_{h\in Q}\right) = 0.$$

*Furthermore, the economy* $\mathbf{E}$ *has social diversity index* $I(\mathbf{E})$ *if and only if* $I(E) = H - K$, *where K satisfies the following conditions*: (i) *for every set* $Q \subset \{1,\dots,H\}$ *of cardinality at most K*

$$\mathbf{H}^*\left(\text{nerve}\{D_h\}_{h\in Q}\right) = 0,$$

[30] With integer coefficients.

*and* (ii) *there exists* $T \subset \{1, \ldots, H\}$ *with cardinality* $T = K + 1$ *and*

$$\mathbf{H}^*\left(\text{nerve}\{D_h\}_{h \in T}\right) \neq 0.$$

*Proof.* This follows from Theorem 5 and Corollary 2 to Theorem 6, p. 200 of Chichilnisky (1993a), which proves that an acyclic family of sets has non empty intersection if and only if every subfamily has acyclic union, [31] see Chichilnisky (1993a) for details. □

## 8. Conclusions

I extended prior results on necessary and sufficient conditions for the existence of a competitive equilibrium in Chichilnisky (1995a) to mixed economies which treat all preferences in a unified way, following Chichilnisky (1995b). I defined social diversity and a topological invariant which contains exact information about the existence of a competitive equilibrium for the economy and its subeconomies, and showed that when limited arbitrage fails every quasiequilibrium has a failure of 'effective' demand.


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## Appendix

*Proposition 3. Under Assumption 2:* (i) *The cone* $G_h(\Omega_h)$ *equals* $A_h(\Omega_h)$ *when indifferences contain half lines* (*case* (a)) *and its closure when they do not,* (*case* (b)), *and thus it is identical to the global cone defined in Chichilnisky* (1995*a*); (ii) *the cones* $\bar{A}_h(\Omega_h) = \bar{G}_h(\Omega_h)$ *are independent of initial endowments* $\Omega_h \in R^N$; (iii) *when preferences are in cases* (a) *and* (b) *the global cones* $G_h(\Omega_h)$ *are independent of initial endowments, but they are not necessarily independent in general*; *and* (iv) *none of these results need hold when Assumption* 2 *is not satisfied.*

[31] This result applies also to more general families.

*Proof.* See also Chichilnisky (1995b). Define the sets:

$$B_h(\Omega_h) = \Big\{ x \in X : u_h(\Omega_h + \lambda x) \geq u_h(\Omega_h + \eta x) \text{ when } \lambda \geq \eta \geq 0,$$

$$\lim_{\lambda_j \to \infty} u_h(\Omega_h + \lambda_j x) = u^0 < \infty \text{ and } \exists_j : u_h(\Omega_h + \lambda_j x) = u^0 \Big\},$$

and

$$C_h(\Omega_h) = \Big\{ x \in X : u_h(\Omega_h + \lambda x) \geq u_h(\Omega_h + \eta x) \text{ when } \lambda \geq \eta \geq 0,$$

$$\lim_{\lambda_j \to \infty} u_h(\Omega_h + \lambda_j x) = u^0 < \infty \text{ and } \sim \exists j : u_h(\Omega_h + \lambda_j x) = u^0 \Big\}.$$

The three sets $A_h(\Omega_h)$, $B_h(\Omega_h)$ and $C_h(\Omega_h)$ are disjoint pairwise and

$$A_h(\Omega_h) \cup B_h(\Omega_h) \cup C_h(\Omega_h) \cup H_h(\Omega_h) = R^N, \tag{7}$$

where $H_h(\Omega_h)$ is the complement of the set $A_h(\Omega_h) \cup B_h(\Omega_h) \cup C_h(\Omega_h)$. Observe that by convexity of preferences $H_h(\Omega_h)$ is the set of directions along which the utility achieves a maximum value and decreases thereafter.

The first step is to observe that if $z \in B_h(\Omega_h) \cup C_h(\Omega_h)$, then for all $s \in R^N$

$$s \gg z \Rightarrow s \in A_h(\Omega_h) \tag{8}$$

and

$$s \ll z \Rightarrow s \in H_h(\Omega_h). \tag{9}$$

This follows from monotonicity and Assumption 2, which implies that the rate of increase of the utility is uniformly bounded below above zero along the direction defined by any strictly positive vector. Since the utility $u_h$ is non-satiated (8) and (9) together imply that for every $\varepsilon > 0$, $\exists \lambda > 0$ s.t. an $\varepsilon$ neighborhood of a vector $\lambda z \in B_h(\Omega_h) \cup C_h(\Omega_h)$ contains a vector $s$ in the set $A_h(\Omega_h)$ and another vector $v$ in the set $H_h(\Omega_h)$. This implies that the set $B_h(\Omega_h) \cup C_h(\Omega_h)$ is in the boundary of the set $A_h(\Omega_h)$. The relation between $G_h(\Omega_h)$ and $A_h(\Omega_h)$ stated in (i) is now immediate, cf. Chichilnisky (1995a, p. 85 (4)).

The next step is to show that if two different half-lines $l = \{\Omega_h + \lambda v\}_{\lambda \geq 0}$ and $m = \{\Lambda_h + \lambda v\}_{\lambda \geq 0}$ are parallel translates of each other, and $l \subset A_h(\Omega_h)$, then $m \subset A_h(\Lambda_h)$, $\forall \Lambda_h \in m$. This is immediate from Assumption 2, which ensures that the rate of increase of the utility is uniformly bounded above. Therefore the cone $A_h$ is independent of the initial endowments.

Observe that for a general convex preference represented by a utility $u_h$ the set $G_h(\Omega_h)$ itself may vary as the vector $\Omega_h$ varies, since the set $B_h(\Omega_h)$ itself may vary with $\Omega_h$: at some $\Omega_h$ a direction $z \in \partial G_h$ may be in $B_h(\Omega_h)$ and at others $B_h(\Omega_h)$ may be empty and $z \in C_h(\Omega_h)$ instead. This occurs when along a ray defined by a vector $z$ from one endowment the utility levels asymptote to a finite limit but do not reach their limiting value, while at other endowments, along the same direction $z$, they achieve this limit. This cannot happen when the preferences are in cases (a) or (b), but is otherwise consistent with Assumption 2. This example, and similar reasonings for $A_h(\Omega_h)$ completes the proof of the proposition. □

*Proof of Theorem 1* (Limited arbitrage is equivalent to the compactness of $U(\mathbf{E})$). This result always holds when the consumption set is bounded below by some vector in the space, and in that case it is proved using standard arguments; see, for example, Arrow and Hahn (1971). Therefore in what follows I concentrate in the case where $X$ is unbounded below.

Sufficiency first. Assume $\mathbf{E}$ has limited arbitrage. Since global cones are independent of initial endowments by Proposition 3, we may assume without loss of generality that $\Omega_h = 0$ for all $h$. If $U(\mathbf{E})$ was *not* bounded there would exist a sequence of individually rational net trades $(z_1^j \dots z_H^j)_{j=1,2\dots}$ and therefore correspondingly positive utility levels, such that: $\forall_j, \sum_h z_h^j = 0$, and for some $h$, $\lim_{j\to\infty} u_h(z_h^j) = \infty$. Therefore for some $h$, $\lim_j \| z_h^j \| = \infty$, and there exists a subsequence of its normalized vectors, denoted also by $\{z_h^j / \| z_h^j \|\}$, which is convergent. I will now prove that $z_h = \lim_j \; z_h^j / \| z_h^j \|$ is in $\overline{G}_h$, the closure of $G_h$. The proof is by contradiction. If $z_h \notin \overline{G}_h$ then by quasiconcavity of $u_h$ and by the proof of Proposition 3, along the ray defined by $z_h$ the utility $u_h$ achieves a maximum level $u^0$ at $\lambda_0 z$, for some $\lambda_0 \geq 0$, and decreases thereafter: $\lambda > \lambda_0 \Rightarrow u_h(\lambda z_h) < u^0$. Define a function $\theta: R_+ \to R_+$ by $u_h(\lambda z_h + \theta(\lambda) e) = u^0$, where $e = (1, \dots, 1)$. I will show that $\theta$ is a convex function. By the convexity of preferences,

$$u^0 \leq u_h\Big(\alpha\big(\lambda z_h + \theta(\lambda) e\big) + (1-\alpha)\big(\hat{\lambda} z_h + \theta(\hat{\lambda}) e\big)\Big) \tag{10}$$

$$= u_h\Big(\big(\alpha\lambda + (1-\alpha)\hat{\lambda}\big) z_h + \big(\alpha\theta(\lambda) + (1-\alpha)\big(\theta(\hat{\lambda})\big)e\Big). \tag{11}$$

Thus by monotonicity, $\theta(\alpha\lambda + (1-\alpha)\hat{\lambda}) \leq \alpha\theta(\lambda) + (1-\alpha)\theta(\hat{\lambda})$, which proves convexity.

Assumption 2 together with monotonicity implies that the rate of increase of $u_h$ along the direction defined by $e$ (or by any strictly positive vector) is uniformly bounded below: $\exists \varepsilon > 0 : | u_h(x + \theta e) - u_h(x) | \geq | \theta | \varepsilon$, $\forall \theta \in R_+$, $\forall x \in R^N$. Therefore $u_h(\lambda z_h + \theta(\lambda) e) \equiv u^0 \geq u_h(\lambda z_h) + \theta(\lambda)\varepsilon$, so that $u_h(\lambda z_h) \leq u^0 - \theta(\lambda)\varepsilon$. Note that $\theta(\lambda_0) = 0$ and $\theta(\lambda) > 0$ for $\lambda > \lambda_0$. Since as seen above $\theta$ is convex, $\lim_{h\to\infty} \; \theta(\lambda) = \infty$, and $u_h(\lambda z_h) \leq u^0 - \theta(\lambda)\varepsilon$ implies $u_h(\lambda z_h) \to -\infty$. It follows that $z_h \in \overline{G}_h$ for otherwise, since $\lim_j \| z_h^j \| = \infty$ and the vectors $z_h^j$ wander arbitrarily close to the direction defined by $z_h$, $\lim_{j\to\infty} \; u_h(z_h^j) < 0$ contradicting individually rationality of the allocations $(z_1^j \dots z_H^j)_{j=1,2\dots}$

Now recall that for some $g$ $\lim_{j\to\infty} \; u_g(z_g^j) = \infty$. By Assumption 2, $| u_g(x) - u_g(y) | \leq K \| x - y \| \forall x,\; y \in R^N$, so for any $n$ and $j \,| u_g(z_g^n) - u_g(z_g^n - je) | \leq$

$K \| je \|$; therefore for every $j$ there exists an $n_j$ such that $u_g(z_g^{n_j} - je) > j$. Take the sequence $\{z_g^{n_j}\}$ and relabel it $\{z_g^j\}$. Now consider the new sequence of allocations $\{z_1^j + je/(H-1), \dots, z_g^j - je, \dots, z_H^j + je/(H-1)\}$ and call it also $\{z_h^j\}_{h=1,2,\dots,H}$; this defines a feasible allocation for all $j$ and by Assumption 2, $\forall h,\ u_h(z_h^j) \to \infty$. In particular $\forall h,\ \| z_h^j \| \to \infty$. Define now $C$ as the set of all strictly positive convex combinations of the vectors $z_h$ for all $h$. Then either $C$ is strictly contained in a half space, or it defines a subspace of $R^N$. Since $\Sigma_h z_h^j = 0$, $C$ cannot be strictly contained in a half space. Therefore, $C$ defines a subspace; in particular for any $g$, $\exists \lambda_h \geq 0, \forall h \neq g$, such that

$$-z_g = \sum_h \lambda_h z_h. \qquad (*)$$

When one trader $g$ has indifferences without half lines, case (b), then $G_g = \overline{G}_g$ and therefore $z_g \in \overline{G}_g \Rightarrow z_g \in G_g$, which by $(*)$ contradicts limited arbitrage because $\sim \exists p : \langle p, z_h \rangle \geq 0$ for $z_h \in \overline{G}_h$ all $h$, and $\langle p, z_g \rangle > 0\, z_g \in G_g$. When for all $h$ the normalized gradients to any closed set of indifferent vectors define a closed set, case (a), the global cone $G_h$ is open (Chichilnisky, 1995a) so that its complement $G_h^c$ is closed, and the set of directions in $G_h^c$ is compact. On each direction of $G_h^c$ the utility $u_h$ achieves a maximum; therefore there exists for each $h$ a maximum utility level for $u_h$ over all directions in $G_h^c$. Since along the sequence $\{z_h^j\}$ every trader's utility increases without bound, $\forall h \exists j_h : j > j_h \Rightarrow z_h^j \in G_h$. However $\forall j,\ \Sigma_h z_h^j = 0$, again contradicting limited arbitrage. In all cases a contradiction arises from assuming that $U(\mathbf{E})$ is not bounded, so $U(\mathbf{E})$ must be bounded.

The next step is to prove that the set $U(\mathbf{E})$ is closed when limited arbitrage is satisfied. Consider a sequence of allocations $\{z_h^j\}_{j=1,2\dots}$, $h = 1, 2, \dots, H$, satisfying $\forall_j,\ \Sigma_{h=1}^H z_h^j = 0$. Assume that $\forall j,\ u_1(z_1^j), \dots, u_H(z_H^j) \subset R_+^H$ and converges as $j \to \infty$ to a utility allocation $v = (v_1, \dots, v_H) \in R_+^H$, which is undominated by any other feasible utility allocation. Observe that the vector $v$ may or may not be the utility vector of a feasible allocation: when limited arbitrage is satisfied, I will prove that it is. The result is immediate if the sequence of allocations $\{z_h^j\}_{j=1,2\dots}$, $h = 1, 2, \dots, H$ is bounded; therefore I concentrate in the case where the set of feasible allocations is not bounded. Let $M$ be the set of all traders $h \in \{1, 2.,., H\}$, which may be empty, for whom the corresponding sequence of allocations $\{z_h^j\}_{j=1,2\dots}$ is bounded, i.e. $h \in M \Leftrightarrow \exists K_h : \forall j,\ \| z_h^j \| < K_h < \infty$; let $J$ be its complement, $J = \{1, 2, \dots, H\} - M$, which I assume to be non empty. There exists a subsequence of the original sequence of allocations, which for simplicity is denoted also $\{z_h^j\}_{j=1,2\dots}$, $h = 1, 2, \dots, H$, along which $\forall h \in M$, $\lim_j \{z_h^j\}_{j=1,2\dots} = z_h$ exists, and by construction $\Sigma_{h \in M} z_h + \lim_{j \to \infty} \Sigma_{h \in J} z_h^j = 0$. Consider now the sequence $z_h^j / \| z_h^j \|$ for $h \in J$; it has a convergent subsequence, denoted also $z_h^j / \| z_h^j \|$. Define $z_h = \lim_j z_h^j / \| z_h^j \|$, $h \in J$. Then as seen in the first part of this proof, $\forall h \in J,\ z_h \in \overline{G}_h$, because $\| z_h^j \| \to \infty$ and $u_h(z_h^j) \geq 0$. If $\forall h \in J,\ z_h \notin G_h$, then the utility values of the traders attain their limit for all $h$

and the utility vector $v$ is achieved by a feasible allocation. Therefore the proof is complete. It only remains to consider the case where for some trader $g \in J$, $z_g \in G_g$. As above, let $C$ be the convex cone of all strictly positive linear combinations of the vectors $\{z_h\}_{h \in J}$. Then either $C$ is contained strictly in a half-space of $R^N$, or $C$ spans a subspace of $R^N$. Since $\Sigma_{h \in M} z_h + \lim_{j \to \infty} \Sigma_{h \in J} z_h^j = 0$, $C$ cannot be strictly contained in a half space, for otherwise $\lim_{j \to \infty} \Sigma_{h \in J} z_h^j$ would not be bounded as $\forall h \in J$, $\| z_h^j \| \to \infty$. Therefore there exists a subspace $S \subset R^N$ spanned by $\{z_h\}_{h \in J}$. In particular, $-z_g \in S$, i.e. $\forall h \in J, \exists \lambda_h \geq 0$ such that $(^*) -z_g = \Sigma_{h \in J} \lambda_h z_h$. Since in this last case $\forall h \in J$, $h \neq g$, $z_h \in \overline{G}_h$ and for $h = g$, $z_g \in G_g$, by limited arbitrage $\exists p \in \bigcap_h D_h$ s. t. $\langle p, z_g \rangle > 0$, and $\forall h$, $\langle p, z_h \rangle \geq 0$, which contradicts $(*)$. Since the contradiction arises from assuming that the set $U(\mathbf{E})$ is not closed, $U(\mathbf{E})$ must be closed. Thus $U(\mathbf{E})$ is compact. In particular, since $P(\mathbf{E})$ is the boundary of $U(\mathbf{E})$, limited arbitrage implies a compact Pareto frontier $P(\mathbf{E})$.

I establish necessity next. If limited arbitrage fails, there is no vector $y \in R^N$ such that $\langle y, z_h \rangle > 0$ for all $\{z_h\} \in G_h$. Equivalently, there exist a set $J$ consisting of at least two traders and, for each $h \in J$, a vector $z_h \in G_h$ such that $\Sigma_{h \in J} z_h = 0$. Then either for some $h$, $z_h \in A_h$ so that the Pareto frontier is unbounded and therefore not compact, or else for some $h$, $z_h \in \partial G_h \cap G_h$ and therefore the Pareto frontier is not closed, and $U(\mathbf{E})$ is not compact. In either case, the Pareto frontier is not compact when limited arbitrage fails. Therefore compactness of $U(\mathbf{E})$ is necessary for limited arbitrage. □

*Proof of Theorem 2* (Limited arbitrage is equivalent to the existence of a competitive equilibrium). Necessity first. Consider first the case $X = R^N$ and assume without loss of generality that $\Omega_h = 0$ for all $h$. The proof is by contradiction. Assume that limited arbitrage fails and let $p^*$ be an equilibrium price and $x^* = (x_1^*, \ldots x_H^*)$ the corresponding equilibrium allocation. A failure of limited arbitrage means that $\exists\ (z_1, \ldots, z_H) : \forall h,\ z_h \in G_h$, and $\Sigma_{h=1}^H z_h = 0$. By construction $u_h(\lambda z_h)$ never ceases to increase with $\lambda$, because $z_h \in G_h$. Since by Proposition 3 global cones are uniform $x_h + z_h \in G_h(x_h^*)$, so that by the same reasoning $x^*$ cannot be Pareto efficient, contradicting the fact that $x^*$ is a competitive equilibrium.

Consider next $X = R_+^N$. Assume that $\forall q \in S(\mathbf{E}) \exists h \in \{1, \ldots, H\}$ such that $\langle q, \Omega_h \rangle = 0$. Then if limited arbitrage is not satisfied $\bigcap_{h=1}^H D_h^+(\Omega_h) = \emptyset$, which implies that $\forall p \in S(\mathbf{E})$, $\exists h$ and $v(p) \in G_h(\Omega_h)$:

$$\langle p, \lambda v(p) \rangle \leq 0, \quad \forall \lambda > 0. \tag{12}$$

I will show that this implies that a competitive equilibrium price cannot exist: the proof is by contradiction. Let $p^*$ be an equilibrium price and $x^* \in X^H$ be the corresponding equilibrium allocation. Then $p^* \in S(E)$, so that by (12) $x_h^* + \lambda v(p)$ is affordable and strictly preferred to $x^*$ for some $\lambda > 0$, contradicting the assumption that $x^*$ is an equilibrium allocation. Therefore limited arbitrage is also necessary for the existence of a competitive equilibrium in this case.

It remains to consider the case where $\exists p \in S(\mathbf{E})$ such that $\forall h \in \{1,\ldots,H\}$, $\langle p, \Omega_h \rangle \neq 0$. But in this case by definition $\bigcap_{h=1}^{H} D_h^+(\Omega_h) \neq \emptyset$ since $\forall h \in \{1\ldots H\}$, $R^N_{++} \subset D_h^+(\Omega_h)$, so that limited arbitrage is always satisfied when an equilibrium exists. This completes the proof of necessity when $X = R^N_+$.

Sufficiency next. The proof uses the fact that the Pareto frontier is homeomorphic to a simplex. When $X = R^N_+$ the Pareto frontier of the economy $P(\mathbf{E})$ is always homeomorphic to a simplex, see Arrow and Hahn (1971). In the case $X = R^N$ this may fail: for example $U(\mathbf{E})$ may be unbounded. However, by Theorem 1 above, if the economy satisfies limited arbitrage then the utility possibility set $U(\mathbf{E}) \subset R^N_+$ and the Pareto frontier $P(\mathbf{E}) \subset R^N_+$ are compact; under the assumptions on preferences, $P(\mathbf{E})$ is then also homeomorphic to a simplex (Arrow and Hahn, 1971). Therefore in both cases, $P(\mathbf{E})$ is homeomorphic to a simplex and I can apply a method due to Chichilnisky and Heal (1984) which extends Negishi's method of using a fixed point argument on the Pareto frontier [32] to establish the existence of a *pseudoequilibrium* with or without short sales. [33] It remains, however, to prove that the pseudoequilibrium is also a competitive equilibrium.

To complete the proof of existence of a competitive equilibrium consider first $X = R^N$. Then $\forall h = 1,\ldots,H$ there exists an allocation in $X$ of strictly lower value than the pseudoequilibrium $x_h^*$ at the price $p^*$. Therefore by Lemma 3, Chapter 4, page 81 of Arrow and Hahn (1971), the quasiequilibrium $(p^*, x^*)$ is also a competitive equilibrium, completing the proof of existence when $X = R^N$.

Next consider $X = R^N_+$, and a quasi-equilibrium $(p^*, x^*)$ whose existence was already established. If every individual has a positive income at $p^*$, i.e. $\forall h, \langle p^*, \Omega_h \rangle > 0$, then by Lemma 3, Chapter 4 of Arrow and Hahn (1971) the quasiequilibrium $(p^*, x^*)$ is also a competitive equilibrium, completing the proof. Furthermore, observe that in any case the pseudoequilibrium price $p^* \in S(\mathbf{E})$, so that $S(\mathbf{E})$ is not empty. To prove existence we consider therefore two cases: first the case where $\exists q^* \in S(\mathbf{E}) : \forall h, \langle q^*, \Omega_h \rangle > 0$. In this case, by the above remarks from Arrow and Hahn (1971), $(q^*, x^*)$ is a competitive equilibrium. The second case is when $\forall q \in S(\mathbf{E})$, $\exists h \in \{1,\ldots,H\}$ such that $\langle q, \Omega_h \rangle = 0$. Limited arbitrage then implies:

$$\exists q^* \in S(\mathbf{E}) : \forall h, \langle q^*, v \rangle > 0 \quad \text{for all } v \in G_h(\Omega_h). \tag{13}$$

Let $x^* = x_1^*, \ldots, x_H^* \in X^H$ be a feasible allocation in $\Upsilon$ supported by the vector

[32] Negishi's method for proving the existence of a pseudoequilibrium (Negishi, 1960) applies only to the case where the economy has no short sales. This result was extended by Chichilnisky and Heal (1984, 1993) to economies with short sales, where feasible desirable allocations may be unbounded; see also Chichilnisky and Heal (1991b) and Chichilnisky (1995a).

[33] A pseudoequilibrium, also called quasiequilibrium, is an allocation and a price at which traders minimize cost and markets clear.

$q^*$ defined in (13): by definit… , $\forall h$, $u_h(x_h^*) \geq u_h(\Omega_h)$ and $q^*$ supports $x^*$. Note that any $h$ minimizes costs at $x_h^*$ because $q^*$ is a support. Furthermore $x_h^*$ is affordable under $q^*$. Therefore, $(q^*, x^*)$ can fail to be a competitive equilibrium only when for some $h$, $\langle q^*, x_h^* \rangle = 0$, for otherwise the cost minimizing allocation is always also utility maximizing in the budget set $B_h(q^*) = \{w \in X : \langle q^*, w \rangle = \langle q^*, \Omega_h \rangle\}$.

It remains therefore to prove existence when $\langle q^*, x_h^* \rangle = 0$ for some $h$. Since by the definition of $S(\mathbf{E})$, $x^*$ is individually rational, i.e. $\forall h$, $u_h(x_h^*) \geq u_h(\Omega_h)$, then $\langle q^*\ x_h^* \rangle = 0$ implies $\langle q^*, \Omega_h \rangle = 0$, because by definition $q^*$ is a supporting price for the equilibrium allocation $x^*$. If $\forall h$, $u_h(x_h^*) = 0$ then $x_h^* \in \partial R_+^N$, and by the monotonicity and quasiconcavity of $u_h$, any vector $y$ in the budget set defined by the price $p^*$, $B_h(q^*)$, must also satisfy $u_h(y) = 0$, so that $x_h^*$ maximizes utility in $B_h(q^*)$, which implies that $(q^*, x^*)$ is a competitive equilibrium. Therefore $(q^*, x^*)$ is a competitive equilibrium unless for some $h$, $u_h(x_h^*) > 0$.

Assume therefore that the quasiequilibrium $(q^*, x^*)$ is not a competitive equilibrium, and that for some $h$ with $\langle q^*, \Omega_h \rangle = 0$, $u_h(x_h^*) > 0$. Since $u_h(x_h^*) > 0$ and $x_h^* \in \partial R_+^N$ then an indifference surface of a commodity bundle of positive utility $u_h(x_h^*)$ intersects $\partial R_+^N$ at $x_h^* \in \partial R_+^N$. Let $r$ be the ray in $\partial R_+^N$ containing $x_h^*$. If $w \in r$ then $\langle q^*, w \rangle = 0$, because $\langle q^*, x_h^* \rangle = 0$. Since $u_h(x_h^*) > 0$, by Assumption 1 $u_h$ does not reach a maximum along $r$, so that $w \in G_h(x_h^*)$. But this contradicts the choice of $q^*$ as a supporting price satisfying limited arbitrage (13) since

$$\exists h \text{ and } w \in G_h(\Omega_h) \text{ such that } \langle q^*, w \rangle = 0. \tag{14}$$

The contradiction between (14) and (13) arose from the assumption that $(q^*, x^*)$ is not a competitive equilibrium, so that $(q^*, x^*)$ must be a competitive equilibrium, and the proof is complete. □

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